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AN INFINITE SEQUENCE OF $\Gamma\Delta$-REGULAR GRAPHS

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1. Introduction

In this paper we are interested in graphs which, in a sense, are a generalization of *strongly regular graphs*. We remind the reader that a strongly regular graph with parameters n, k, λ, μ (notation SRG(n, k, λ, μ)) is a graph on n vertices, regular of degree k, and such that any two vertices joined, resp. not joined, by an edge have λ, resp. μ, common neighbours. If G is a graph and x a vertex of G, then $\Gamma(x)$ will denote the set of neighbours of x and also the induced subgraph on these vertices. Similarly with $\Delta(x)$ for the non-neighbours. So, in an SRG(n, k, λ, μ) both $\Gamma(x)$ and $\Delta(x)$ are regular subgraphs (with degree λ, resp. $k-\mu$). The following problem was suggested by Seidel. Study the class of graphs with the property that $\Gamma(x)$ and $\Delta(x)$ are *regular* for every vertex x of G. Notice that no requirement is made about the degree of the subgraphs $\Gamma(x)$ and $\Delta(x)$. We call such a graph G a *neighbourhood-regular graph* or $\Gamma\Delta$-regular graph. In 1979 these graphs were studied by Godsil and McKay [2]. To give the reader some feeling for the problem we briefly survey their most important results.

Lemma 1.1. *If G is connected and $\Gamma(x)$ is regular for every $x \in G$, then there is a number λ such that each $\Gamma(x)$ has degree λ.*

Proof. Let $x \sim y$. Then the degree of $\Gamma(x)$ is $|\Gamma(x) \cap \Gamma(y)|$. This is symmetric in x and y. Since G is connected, we are done. $\square$

Corollary 1.2. *If G is $\Gamma\Delta$-regular and both G and $\bar{G}$ are connected, then there are numbers λ and $\bar{\lambda}$ such that $\Gamma(x)$ has degree λ and $\overline{\Delta(x)}$ has degree $\bar{\lambda}$.*

From this it easily follows that if a $\Gamma\Delta$-regular graph is regular, then it is a strongly regular graph. It is also easy to check that if a $\Gamma\Delta$-regular graph G is disconnected and non-regular, then G is the disjoint union of two complete graphs or $G = K_m \cup G'$ where $G' = \mathrm{SRG}(v, k, \lambda, \mu)$ and $m = k - \mu + 1$. From now on we are only interested in $\Gamma\Delta$-regular graphs G which are non-regular and

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such that both G and $\bar{G}$ are connected. We call these graphs *non-trivial* $\Gamma\Delta$-regular graphs. Besides the parameters n, λ, $\bar{\lambda}$ associated with G, there are several other parameters which were obtained in [2]. We only state these below. The proofs all depend on not too difficult counting arguments.

Theorem 1.3. *In a non-trivial $\Gamma\Delta$-regular graph with parameters n, λ, $\bar{\lambda}$ the following holds*:

(i) *There are two numbers k_1, $k_2 (k_1 < k_2)$ such that each $x \in G$ has degree k_1 or k_2; we define $\eta := k_2 - k_1$.*

(ii) *If $u \in G$, $v \in G$, $u \not\sim v$, then*

$$|\Gamma(u) \cap \Gamma(v)| = \begin{cases} \lambda + 1 & \text{if } \deg(u) \neq \deg(v), \\ \lambda + 1 + \eta & \text{if } \deg(u) = \deg(v) = k_2, \\ \lambda + 1 - \eta & \text{if } \deg(u) = \deg(v) = k_1. \end{cases}$$

(iii) *Let $M_i := \{x \in G \mid \deg(x) = k_i\}$, $m_i := |M_i|$. There are numbers α_1, α_2 such that $|\Gamma(x) \cap M_i| = \alpha_i$ for all x in M_i.*

(iv) $\lambda + \bar{\lambda} = \dfrac{n}{2} - 2$ (*and hence n must be even*).

(v) $k_1 + k_2 = \dfrac{n}{2} + 2\lambda + 1$.

2. Algebraic formulation

A number of the non-existence results given in [2] depend on studying the eigenvalues of the adjacency matrix A of G. We can write A in the form

$$A = \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix},$$

where A_i is the adjacency matrix of M_i. Then (1.2) and (1.3) yield the following equations:

$$\begin{aligned} A_1^2 + BB^T &= (k_2 - \lambda - 1)I + (\lambda + 1 - \eta)J + (\eta - 1)A_1, \\ A_2^2 + B^TB &= (k_1 - \lambda - 1)I + (\lambda + 1 + \eta)J - (\eta + 1)A_2, \\ A_1B + BA_2 &= (\lambda + 1)J - B. \end{aligned} \tag{2.1.}$$

These equations make it possible to prove statements about the eigenvalues of A, which then restrict the possibilities for the submatrices M_1 and M_2 (see [2] and [3] for a number of non-existence results obtained by this approach). If A is the $(0, 1)$ adjacency matrix of a graph, then the $(0, +1, -1)$-matrix

$$N := J - I - 2A \tag{2.2.}$$

is called the $(0, +1, -1)$-matrix of the graph. The equations (2.1.) can be rewritten in terms of the $(0, +1, -1)$-matrix of the $\Gamma\Delta$-regular graph.

Lemma 2.3. *Let G be a non-trivial ΓΔ-regular graph with a* $(0, +1, -1)$*-matrix N of the following form*:

$$N = \begin{pmatrix} N_1 & M \\ M^T & N_2 \end{pmatrix},$$

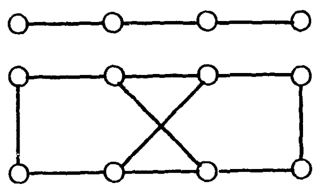
where N_i *is the* $(0, +1, -1)$*-matrix of* $M_i (i = 1, 2)$. *Then*:

$$N_1^2 + MM^T = (n-1)I - 2\eta N_1,$$
$$N_2^2 + M^TM = (n-1)I + 2\eta N_2,$$
$$N_1M + MN_2 = O.$$

Proof. This follows from (2.1) and Theorem (1.3)(i), (v). □

3. Some examples

Until now non-trivial $\Gamma\Delta$-regular graphs on n vertices were known only for $n \in \{4, 8, 28, 32\}$, (cf. [2]). The unique graphs on 4 resp. 8 vertices are



$k_1 = 1,\ k_2 = 2, \qquad m_1 = m_2 = 2;$

$k_1 = 2,\ k_2 = 3, \qquad m_1 = m_2 = 4.$

We only mention one more example. It has adjacency matrix

$$A = \begin{pmatrix} A_1 & A_1 + I \\ A_1 + I & J - A_1 - I \end{pmatrix}.$$

where A_1 is the adjacency matrix of a $(16, 6, 2, 2)$ strongly regular graph (e.g. $L_2(4)$). In each of these examples $m_1 = m_2$ and furthermore the first two also have $\eta = 1$. These two are the first two examples of the sequence we shall construct in Section 5.

4. The case $m_1 = m_2$ and $\eta = 1$

If G is a $\Gamma\Delta$-regular graph with $m_1 = m_2$ and $\eta = 1$, with $(0, +1, -1)$ matrix N as in (2.3), then we introduce the matrix

$$H := \begin{pmatrix} N_1 + I & M \\ M^T & N_2 - I \end{pmatrix}.$$

Lemma 4.1. *The matrix H is a Hadamard matrix of order n.*

Proof. In fact (2.3) implies

$$\begin{pmatrix} N_1 + \eta I & M \\ M^T & N_2 - \eta I \end{pmatrix}^2 = (n - 1 + \eta^2)I. \quad \square$$

Of course, we also know that H is a *symmetric* matrix: a well-known construction of symmetric Hadamard matrices makes use of a symmetric *conference* matrix C (cf. [4]).

Lemma 4.2. *If C is a symmetric conference matrix then*

$$H_1 := \begin{pmatrix} I+C & I-C \\ I-C & -I-C \end{pmatrix}$$

is a symmetric Hadamard matrix.

A closer analysis of the requirements (2.1) (in their $(0, +1, -1)$-form) leads to the following idea. Let $\tilde{C}$ be a symmetric conference matrix. Consider

$$N_1 = -N_2 = \begin{pmatrix} I+\tilde{C} & I-\tilde{C} \\ I-\tilde{C} & I+\tilde{C} \end{pmatrix}, \qquad M = \begin{pmatrix} I+\tilde{C} & -I+\tilde{C} \\ -I+\tilde{C} & I+\tilde{C} \end{pmatrix}.$$

Then

$$H := \begin{pmatrix} N_1 & M \\ M^T & N_2 \end{pmatrix}$$

is indeed a symmetric Hadamard matrix and N_1 has diagonal $+1$, N_2 has diagonal -1. If we now check the relations (2.1), it turns out that we have constructed a $\Gamma\Delta$-regular graph if the conference matrix $\tilde{C}$ is *regular,* i.e. $\underline{j}$ is eigenvector of $\tilde{C}$.

5. The construction

Let p be an odd integer. Suppose a projective plane Π of order p exists. Let l be a line in Π and let $\Pi_0 := \Pi \backslash l$ be the corresponding affine plane. We split l into two disjoint sets Ω^+ and Ω^- of $(p+1)/2$ points each. It is well known (cf. [1]) that the net in Π_0 corresponding to the directions of Ω^- corresponds to a strongly regular graph

$$\mathrm{SRG}\left(p^2, \frac{p^2-1}{2}, \frac{p^2-5}{4}, \frac{p^2-1}{4}\right).$$

This is a so-called half-case SRG. It has $(0, +1, -1)$-matrix N_p such that

$$N_p^2 = p^2 I - J, \qquad N_p J = O. \tag{5.1}$$

Therefore

$$C := \begin{pmatrix} 0 & \underline{j}^T \\ \underline{j} & N_p \end{pmatrix} \tag{5.2}$$

is a symmetric conference matrix of order $p^2 + 1$.

We now wish to switch this matrix (cf. [5]) with respect to a subset S of the vertices in such a way that the resulting conference matrix $\tilde{C}$ is regular. To do this, the induced subgraph on S must be a regular graph on $p(p-1)/2$ vertices with degree $(p-3)(p+1)/4$ such that the subgraph on $\Pi_0 \backslash S$ is also regular (of degree $(p^2-1)/4$). We achieve this as follows. Take a point $x \in \Omega^+$ and let S be the points of Π_0 on any subset of $(p-1)/2$ lines through x. It is trivial to check that all requirements are fulfilled. Therefore we have proved the following theorem.

Theorem 5.3. *Let the odd integer p be the order of a projective plane. Then there exists a $\Gamma\Delta$-regular graph with parameters*

$$\begin{aligned}
&m_1 = m_2 = 2(p^2+1)\\
&\eta = 1\\
&\alpha_1 = p^2, \qquad \alpha_2 = p^2+1\\
&k_1 = 2p^2 - p + 1, \qquad k_2 = 2p^2 - p + 2\\
&\lambda = p^2 - p, \qquad \bar{\lambda} = p^2 + p.
\end{aligned}$$

Examples

(a) The degenerate case $p = 1$ leads to the graph on 8 vertices given in Section 3.

(b) For $p = 3$ we find the graph with adjacency matrix

$$\left(\begin{array}{cc|cc}
P & \bar{P} & P & P+I\\
\bar{P} & P & P+I & P\\
\hline
P & P+I & \bar{P} & P+I\\
P+I & P & P+I & \bar{P}
\end{array}\right),$$

where P is the adjacency matrix of the Petersen graph, and $\bar{P} = J - I - P$.

Remarks

The results given above are part of the author's master's thesis [3]. This thesis contains several non-existence results and also a second infinite class of $\Gamma\Delta$-regular graphs. The material in this paper was presented at the Oberwolfach meeting on Combinatorics in January 1986 by my supervisor, Prof. J. H. van Lint. I thank him for writing this paper for me and I also thank Dr. A. Blokhuis and Dr. H. A. Wilbrink for their help and encouragement during the preparation of the thesis.

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